ECON 4113. HOMEWORK 2 (100 POINTS).
DUE TUESDAY FEBRUARY 22 IN CLASS.

In your answer you may skip long derivations. For example, when calculating the rank of a matrix you can just say what it is, without providing the calculations necessary to determine that. The same holds for the things you do with the systems of equations: just write the relationship between the variables that you think is important.

1. (30 points) Solve using the Lagrange Theorem

$$\begin{aligned} \max_{x,y,z\in\mathbb{R}} \quad & xyz + z \\ s.t. \quad & x^2 + y^2 + z \leq 6 \\ & x, y, z \geq 0 \end{aligned}$$

2. (30 points) Solve using the Lagrange Theorem

$$\begin{aligned} \max_{x,y\in\mathbb{R}} \quad & 3xy - x^3 \\ s.t. \quad & 2x - y = -5 \\ & 5x + 2y \geq 37 \\ & x, y \geq 0 \end{aligned}$$

3. (40 points) Facts: if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then the set $\{x \in \mathbb{R}^n : f(x) \leq 0\}$ is closed and $\{x \in \mathbb{R}^n : f(x) < 0\}$ is open. Also for any finite number of sets $S_1, ..., S_k$ it is true that: 1) if all S_i are closed then the intersection $\cap_{i=1..k} S_i$ is closed; 2) if all S_i are open then the intersection $\cap_{i=1..k} S_i$ is open.

   Call $f : \mathbb{R}^n \to \mathbb{R}$ *strictly increasing* if

$$\forall x, y \in \mathbb{R}^n \quad \Big[x \geq y \Rightarrow f(x) > f(y) \Big]$$

   where $x \geq y$ means that $x_i \geq y_i$ for all $i = 1..n$ and $x_j > y_j$ for at least one $j = 1..n$. Consider the problem

$$\begin{aligned} \max_{x\in\mathbb{R}^n} \quad & f(x) \\ s.t. \quad & G(x) \leq 0 \end{aligned}$$

   where $f : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$, f is continuous and strictly increasing. Prove rigorously that if $\Gamma = \{x \in \mathbb{R}^n : G(x) \leq 0\}$ is bounded then 1) the solution exists and 2) any solution $\bar{x}$ is such that $G^i(\bar{x}) = 0$ for at least one $i \in \{1, ..., m\}$ (that is to say, $\bar{x}$ is not interior solution).